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Propagation of Alfvén–gravitational waves in a stratified perfectly conducting flow with transverse magnetic field

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Alfvén–gravitational waves are found to propagate in a Boussinesq, inviscid, adiabatic, perfectly conducting fluid in the presence of a uniform transverse magnetic field in which the mean horizontal velocity U is independent of vertical height z. The governing wave equation is a fourth-order ordinary differential equation with constant coefficients and is not singular when the Doppler-shifted frequency $\Omega_d = 0$, but is singular when the Alfvén frequency $\Omega_A = 0$. If $\Omega_d^2 < \Omega_A^2$ the waves are attenuated by a factor $\exp -[2\Omega_A(N^2-\Omega_d^2)^{\frac{1}{2}} - \Omega_d^2 + \Omega_A^2]z$, which tends to zero as $z \rightarrow \infty$. This attenuation is similar to the viscous attenuation of waves discussed by Hughes & Young (1966). The interpretation of upward and downward propagation of waves is given.

1. Introduction

Recently, Booker & Bretherton (1967) have investigated the hydrodynamic problem of the critical layer for internal gravity waves in a shear flow. Using the linear theory and the normal-mode technique they obtained, for an inviscid Boussinesq liquid, a wave equation

$$w_{zz} + \left[\frac{N^2}{(U-c)^2} - \frac{U_{zz}}{U-c} - k^2\right] w = 0, \tag{1.1}$$

where w is the vertical velocity, U is the basic velocity, c is the horizontal phase velocity, k is the horizontal wavenumber, $N = (-(g/\rho_0)d\rho_0/dz)^{\frac{1}{2}}$ is the Brunt–Väisälä frequency, ρ_0 is the basic density and the suffix z denotes derivatives with respect to z. Equation (1.1) is singular at $\Omega_d = k(U-c) = 0$, where Ω_d is the Doppler-shifted frequency. Booker & Bretherton have shown that there exists a critical level, namely $\Omega_d = 0$, at which waves are absorbed. That is, the Reynolds stress, which is an appropriate measure of the magnitude of the wave, is reduced on the other side by a factor

$$\exp\{-2\pi(J_H - \tfrac{1}{4})^{\frac{1}{2}}\}, \tag{1.2}$$

where J_H is the hydrodynamic Richardson number. In particular they have given clear physical interpretations for upward- and downward-propagating waves.

The effect of viscosity and heat conduction on internal gravity waves in a viscous shear flow has been investigated by Hazel (1967). He has shown that although the wave equation is not singular at $\Omega_d = 0$, there exists a critical level at which waves are absorbed and the transmission coefficient is given by equation (1.2).

The propagation of magnetohydrodynamic waves in homogeneous or non-homogeneous media has been investigated by many authors (for detailed references see MacDonald 1962), but the propagation of magnetohydrodynamic waves in a stratified conducting shear flow in the presence of a magnetic field has not been given much attention. The aim of the present paper is to consider the propagation of internal gravity waves in a perfectly conducting stratified flow in the presence of a uniform magnetic field transverse to the basic flow. The resulting waves are the combination of the Alfvén waves and the gravitational waves. These waves are of particular interest in geophysical and astrophysical problems, especially the study of the earth's core. This problem is also significant in considering the propagation of Alfvén–gravitational waves from the troposphere to the ionosphere.

When the applied magnetic field is transverse to the basic flow a shear flow is not possible since the basic flow has to be uniform throughout the region of interest to satisfy the magnetic induction equation. In this case, the governing differential equation for the wave motion is of order four and is not singular at $\Omega_d = 0$. In the present problem, as in the case of Hazel (1967), though the wave equation is not singular at $\Omega_d = 0$ in his study, there is an attenuation of waves which depends on the vertical co-ordinate z. This attenuation is similar to the effect of finite electrical conductivity and viscosity in a homogeneous medium discussed by Hughes & Young (1966). Since the wave equation is of order four its solution represents four modes of propagation and, following Booker & Bretherton (1967), we can show that the two modes propagate upwards and the other two propagate downwards.

2. Derivation of wave equation

The flow of a conducting fluid and the form of the magnetic field are governed by the modified Navier–Stokes equations with suitable Maxwell field equations. To derive the linearized equations of motion we make the following assumptions.

(i) The motion is two-dimensional, variations being in the x and z directions.

(ii) The fluid is inviscid, perfectly conducting and adiabatic.

(iii) The Boussinesq approximation.

(iv) The Coriolis forces are neglected.

(v) The components (u, w) of the perturbation velocity, corresponding to a basic flow velocity $(U(z), 0)$, and the components (h_x, h_z) of the perturbation magnetic field, the basic field being H_0 = constant in the (vertical) z direction, are so small that

$$\left|u\frac{\partial}{\partial x}+w\frac{\partial}{\partial z}\right| \ll \left|\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right|$$

and

$$\left|h_x\frac{\partial}{\partial x}+h_z\frac{\partial}{\partial z}\right| \ll \left|\frac{\partial}{\partial t}+H_0\frac{\partial}{\partial z}\right|.$$

The properties of the basic flow imply that $U(z)$ has to be a constant to satisfy the magnetic induction equations.

Under these assumptions, the linearized equations of motion become

$$\begin{bmatrix} \rho_0 D & 0 & D_1 & 0 & -\mu H_0 D_2 & 0 \\ 0 & \rho_0 D & D_2 & g & 0 & -\mu H_0 D_2 \\ D_1 & D_2 & 0 & 0 & 0 & 0 \\ 0 & D_2\rho_0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & D_1 & D_2 \\ -H_0 D_2 & 0 & 0 & 0 & D & 0 \\ 0 & -H_0 D_2 & 0 & 0 & 0 & D \end{bmatrix} \begin{bmatrix} u \\ w \\ P \\ \rho \\ h_x \\ h_z \end{bmatrix} = 0, \qquad (2.1)$$

where P is the perturbed total pressure, ρ_0 is the mean density, ρ is the perturbation density and

$$D = \partial/\partial t + UD_1, \quad D_1 = \partial/\partial x, \quad D_2 = \partial/\partial z.$$

By eliminating u, P, ρ, h_x and h_z from (2.1), we obtain the wave equation

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 (w_{xx} + w_{zz}) + N^2 w_{xx} - A^2(w_{zzzz} + w_{xxzz}) = 0, \qquad (2.2)$$

where

$$A = (\mu H_0^2/\rho_0)^{\frac{1}{2}}$$

is the Alfvén velocity, $N = (g\beta)^{\frac{1}{2}} = [-(g/\rho_0)\,d\rho_0/dz]^{\frac{1}{2}}$ is the Brunt–Väisälä frequency and the subscripts indicate differentiation.

The two-dimensional transient disturbance produced by temporary extraneous forces may be represented as the superposition of a continuum of travelling sinusoidal waves:

$$w = Re\left[\frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^{+\infty} dc\, \hat{w}(k,z,c)\, e^{ik(x-ct)}\right]. \qquad (2.3)$$

Then each Fourier component with a well-defined horizontal wavenumber k and phase velocity c has a vertical structure satisfying

$$\Omega_A^2 \frac{d^4\hat{w}}{dz^4} + k^2(\Omega_d^2 - \Omega_A^2)\frac{d^2\hat{w}}{dz^2} + k^4(N^2 - \Omega_d^2)\hat{w} = 0, \qquad (2.4)$$

where $\Omega_A = kA$. This equation is singular when $\Omega_A = 0$ but not when $\Omega_d = 0$. Hence (2.2) will not reduce to the hydrodynamic equation (1.1) of Booker & Bretherton (1967) when magnetic field tends to zero. In the case of viscous flow discussed by Yanowitch (1967) and in the case of viscous heat-conducting flow discussed by Hazel (1967), though there are critical levels at $\Omega_d = 0$, the corresponding governing wave equations are not singular at these levels, as can be observed from (2.4). When the basic magnetic field is parallel to the mean velocity (Rudraiah & Venkatachalappa 1972) a new phenomenon is encountered if the Doppler-shifted frequency equals the Alfvén frequency. In this case the governing wave equation is singular if c coincides with U or $U \pm A$. Of these three critical layers two are magnetic critical layers and the other a hydrodynamic critical layer. If either the kinematic or magnetic viscosity is retained the resulting term of higher order in the governing equation will prevent the solution from diverging. However, in the case of a transverse magnetic field discussed in the present paper the critical layers are not singular and hence the effect of the transverse magnetic field is to control the critical layers. In other words, the effect of a transverse magnetic field is similar to that of the dissipative processes discussed by Hughes & Young (1966).

The solution of (2.4) is

$$\hat{w} = A_1 e^{il_1 z} + A_2 e^{-il_1 z} + A_3 e^{il_2 z} + A_4 e^{-il_2 z}, \tag{2.5}$$

where

$$l_1 = (1/2^{\frac{1}{2}}A)\{\Omega_d^2 - \Omega_A^2 - [(\Omega_d^2 + \Omega_A^2)^2 - 4\Omega_A^2 N^2]^{\frac{1}{2}}\}^{\frac{1}{2}}, \tag{2.6}$$

$$l_2 = (1/2^{\frac{1}{2}}A)\{\Omega_d^2 - \Omega_A^2 + [(\Omega_d^2 + \Omega_A^2)^2 - 4\Omega_A^2 N^2]^{\frac{1}{2}}\}^{\frac{1}{2}}. \tag{2.7}$$

For the sake of definiteness, we settle the branches for l_1 and l_2 by requiring that

$$l_{1i} > 0, \quad l_{2i} > 0 \quad \text{if} \quad c_i > 0, \tag{2.8}$$

where c_i, l_{1i} and l_{2i} are the imaginary parts of c, l_1 and l_2 respectively. The complete spatial distribution of the vertical velocity is

$$w = \hat{w}\, e^{i(kx-kct)}.$$

3. Discussion of the solution

The nature of solution (2.5) will depend on $\Omega_d = 0$ or $\pm\,\Omega_A$. In this section these cases are discussed, separately, in detail.

3.1. *Phase velocity equal to the basic velocity* ($\Omega_d = 0$)

When the Doppler-shifted frequency $\Omega_d = 0$ (equation for heterogeneity), substitution in (2.1) shows that

$$w\, d\rho_0/dz = 0. \tag{3.1}$$

Hence, either $w = 0$ or $d\rho_0/dz = 0$. If $w = 0$, $u = 0$ and hence there is no wave motion, i.e. the waves are completely absorbed by the mean flow. On the other hand, if $d\rho_0/dz = 0$, i.e. no density stratification, there are no internal gravity waves since the Brunt–Väisälä frequency becomes zero. For internal gravity waves the Brunt–Väisälä frequency should be greater than any other frequency.

3.2. *The Doppler-shifted velocity equal to plus or minus the Alfvén velocity* ($\Omega_d = \pm\,\Omega_A$)

When $N^2 > \Omega_d^2 + \Omega_A^2$ (which is the condition for the existence of internal gravity waves), to satisfy (2.8) we take l_1 and l_2 as

$$l_1 = \mp\,(1/2^{\frac{1}{2}}A)\{-[(\Omega_d^2 + \Omega_A^2)^2 - 4\Omega_A^2 N^2]^{\frac{1}{2}}\}^{\frac{1}{2}}, \tag{3.2}$$

$$l_2 = \mp\,(1/2^{\frac{1}{2}}A)\{[(\Omega_d^2 + \Omega_A^2)^2 - 4\Omega_A^2 N^2]^{\frac{1}{2}}\}^{\frac{1}{2}}, \tag{3.3}$$

where the plus and minus signs correspond to Ω_d negative and positive respectively. Equations (3.2) and (3.3) are used to identify terms in the solution (2.5) as upward- and downward-travelling waves. This can be done in the following three ways.

Group velocity approach. From (2.6) and (2.7) we obtain a dispersion relation in the form

$$\sigma = kU \mp \{A^2l^2 + N^2k^2/(k^2+l^2)\}^{\frac{1}{2}}, \tag{3.4}$$

where $$l = l_1, l_2.$$

We take the minus sign when $\Omega_d = kU - \sigma$ is positive and *vice versa*. Since l_1 and l_2 are complex the group velocity is given by $\partial\sigma/\partial l_r$, where $l_r = \operatorname{Re} l_1$ or $\operatorname{Re} l_2$. In either case ($\Omega_d \gtrless 0$), we find that both $\partial\sigma/\partial l_{1r}$ and $\partial\sigma/\partial l_{2r}$ are positive. Thus the first and the third terms in (2.5) represent upward-propagating waves. Similarly, it can be shown that the second and the fourth solutions represent downward-propagating waves.

Energy approach. A second way of understanding the propagation of upward- and downward-travelling waves, represented by (2.5), is by considering the flow of energy. When the energy flows in the positive z direction the waves carrying the energy are upward-travelling waves, regardless of their phase velocities, whereas when the energy flows in the negative z direction the waves are called downward-travelling waves. The total mean rate of working by the fluid below any level on the fluid above is $\overline{Pw}$, where P is the disturbance pressure and the overbar denotes an average over a horizontal wavelength or over a period. From the linearized equation of horizontal momentum, namely

$$(kU-\sigma)u + \frac{kP}{\rho_0} + \frac{i\mu H_0}{\rho_0}\frac{\partial h_x}{\partial z} = 0, \tag{3.5}$$

we have $$\overline{Pw} = -\rho_0\left[\overline{(U-c)uw} + \overline{\frac{i\mu H_0}{\rho_0 k}\frac{\partial h_x}{\partial z}w}\right]. \tag{3.6}$$

It is found that $\overline{Pw}$ is positive for the first and the third terms of the solution and hence wave energy is flowing upwards. In other words, they represent upward-propagating waves. In the case of second and fourth terms $\overline{Pw}$ is negative and hence they represent downward-propagating waves.

Slightly complex phase velocity. A third way of looking at the concept of upward- and downward-propagating waves is by considering c to be slightly complex with $c_i > 0$. In this case $A_1 e^{il_1 z}$ and $A_3 e^{il_2 z}$ tend exponentially to zero as $z \to \infty$ since the imaginary parts of l_1 and l_2 are always positive. Thus the wave amplitude at every point increases with time, but at any particular time the amplitude decrease as z tends to ∞. Changes in amplitude thus move upward. Hence these terms represent upward-travelling waves. The converse is true for the second and fourth terms, which hence correspond to downward-travelling waves.

3.3. *The horizontal wavenumbers l_1 and l_2 real*

The solution when l_1 and l_2 are complex was discussed in §3.2. However, when l_1 and l_2 are pure real, in which case

$$N^2 > \Omega_d^2 > \Omega_A^2, \quad \Omega_d^2 + \Omega_A^2 > 2\Omega_A N, \tag{3.7}$$

we have $$l_1 = \pm(1/2^{\frac{1}{2}}A)\{\Omega_d^2 - \Omega_A^2 - [(\Omega_d^2+\Omega_A^2)^2 - 4\Omega_A^2N^2]^{\frac{1}{2}}\}^{\frac{1}{2}}, \tag{3.8}$$

$$l_2 = \pm(1/2^{\frac{1}{2}}A)\{\Omega_d^2 - \Omega_A^2 + [(\Omega_d^2+\Omega_A^2)^2 - 4\Omega_A^2N^2]^{\frac{1}{2}}\}^{\frac{1}{2}}. \tag{3.9}$$

In order to satisfy (2.8) we take the positive sign when $\Omega_d > 0$ and the negative sign when $\Omega_d < 0$ for l_1 and *vice versa* for l_2. In this case also, as in the previous case, we can interpret the solutions of (2.5) as upward- or downward-propagating waves.

Group velocity approach. Using (3.4) we can write

$$\frac{\partial \sigma}{\partial l_1} = -\frac{l_1}{\Omega_d}\left[A^2 - \frac{N^2k^2}{(k^2+l_1^2)^2}\right]. \tag{3.10}$$

We note that $A^2 - N^2k^2/(k^2+l_1^2)^2$ is always negative and hence the group velocity is always positive. Thus the first term in (2.5) represents an upward-propagating wave and the second solution corresponds to a downward-propagating wave. Also

$$\frac{\partial \sigma}{\partial l_2} = -\frac{l_2}{\Omega_d}\left[A^2 - \frac{N^2k^2}{(k^2+l_2^2)^2}\right]. \tag{3.11}$$

In this case $A^2 - N^2k^2/(k^2+l_2^2)^2$ is always positive. Therefore $\partial\sigma/\partial l_2$ is always positive and hence we can interpret the third term as an upward-propagating wave and the fourth as a downward-propagating wave.

Energy approach. Using (3.6) we have, for the first term,

$$\overline{Pw} = (\rho_0 l_1|A_1|^2/2k^2\Omega_d)\{\Omega_d^2 + \Omega_A^2 + [\Omega_d^2 + \Omega_A^2 - 4\Omega_A^2N^2]^{\frac{1}{2}}\}, \tag{3.12}$$

which is always positive as the terms in the brackets are positive and l_1 and Ω_d have the same sign. Therefore, the first term in (2.5) represents an upward-propagating wave and the second a downward-propagating wave. Similarly, for the third term in (2.5)

$$\overline{Pw} = (-\rho_0 l_2|A_3|^2/2k^2\Omega_d)\{\Omega_d^2 + \Omega_A^2 - [\Omega_d^2 + \Omega_A^2 - 4\Omega_A^2N^2]^{\frac{1}{2}}\}, \tag{3.13}$$

which is always positive. Thus the wave corresponding to the third term in (2.5) propagates energy upwards and that corresponding to the fourth term propagates energy downwards.

Consideration of a slightly complex phase velocity ($c_i > 0$), as in § 3.2, shows that the first and third terms in (2.5) correspond to upward-propagating waves and the second and fourth to downward-propagating waves.

3.4. *Attenuation of waves*

We have the dispersion relation

$$\sigma = kU + \{A^2l^2 + N^2k^2/(k^2+l^2)\}^{\frac{1}{2}}.$$

If k and σ are real, the imaginary part of l provides a measure of the attenuation of waves. When $\Omega_d^2 < \Omega_A^2$, l is always complex and we have

$$l = \frac{1}{2A}\{[2\Omega_A(N^2 - \Omega_d^2)^{\frac{1}{2}} + (\Omega_d^2 - \Omega_A^2)]^{\frac{1}{2}} + i[2\Omega_A(N^2 - \Omega_d^2)^{\frac{1}{2}} - \Omega_d^2 + \Omega_A^2]^{\frac{1}{2}}\}. \tag{3.14}$$

Hence the waves are attenuated by an amount which decreases to zero as $z \to +\infty$. In particular, from (3.14) it follows that when Ω_d^2 approaches Ω_A^2 the attenuation decreases because the imaginary part of (3.14) decreases. When

$\Omega_d^2 > \Omega_A^2$ and $\Omega_d^2 + \Omega_A^2 > 2\Omega_A N$, l is always real and there will be no attenuation of waves. When $\Omega_d^2 > \Omega_A^2$ and $\Omega_d^2 + \Omega_A^2 < 2\Omega_A N$, l is again complex and hence waves will be attenuated. This attenuation, which arises because of the vertical magnetic field, is similar to the viscous attenuation of waves discussed by Hughes & Young (1966).

When there is no density stratification, i.e. $N = 0$,

$$l = \pm ik, \ \pm \Omega_d/A, \tag{3.15}$$

where $lA = \pm \Omega_d$ corresponds to modified Alfvén waves. In the uniform non-conducting stratified flow discussed by Booker & Bretherton (1967) there are two modes of propagation, the vertical wavelengths of which are given by

$$m = \pm \left\{ \frac{N^2}{(U-c)^2} - k^2 \right\}^{\frac{1}{2}}. \tag{3.16}$$

This mode corresponds to internal gravity waves. When $\Omega_d^2 \gg N^2$ the wavelength represented by the first of (3.15) corresponds to the hydrodynamic case of Booker & Bretherton (1967).

When there is density stratification and (perfect) electrical conductivity, there is a coupling between these two modes, which correspond to modified Alfvén and the internal gravity waves. We conclude that the result of this coupling is the propagation of Alfvén–gravitational waves.

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